**PCT** WORLD INTELLECTUAL PROPERTY ORGANIZATION
International Bureau

INTERNATIONAL APPLICATION PUBLISHED UNDER THE PATENT COOPERATION TREATY (PCT)

(51) International Patent Classification 6 : G06F 7/72 — A1

(11) International Publication Number: **WO 99/49386**

(43) International Publication Date: 30 September 1999 (30.09.99)

(21) International Application Number: PCT/CA99/00254

(22) International Filing Date: 25 March 1999 (25.03.99)

(30) Priority Data:
09/047,518 25 March 1998 (25.03.98) US

(71) Applicant *(for all designated States except US)*: CERTICOM CORP. [CA/CA]; Suite 103, 200 Matheson Boulevard, Mississauga, Ontario L5R 3L7 (CA).

(72) Inventors; and

(75) Inventors/Applicants *(for US only)*: VANSTONE, Scott, A. [CA/CA]; 539 Sandbrook Court, Waterloo, Ontario N2T 2H4 (CA). MULLIN, Ronald, C. [CA/CA]; 533 Twin Oaks Crescent, Waterloo, Ontario N2L 4R9 (CA). ANTIPA, Adrian [CA/CA]; 5623 Cortina Crescent, Mississauga, Ontario L4Z 3R3 (CA). GALLANT, Robert [CA/CA]; 4788 Rosebush Road, Mississauga, Ontario L5M 5N1 (CA).

(74) Agents: PILLAY, Kevin et al.; Orange Chari Pillay, Toronto Dominion Bank Tower, Toronto–Dominion Centre, Suite 3600, P.O. Box 190, Toronto, Ontario M5K 1H6 (CA).

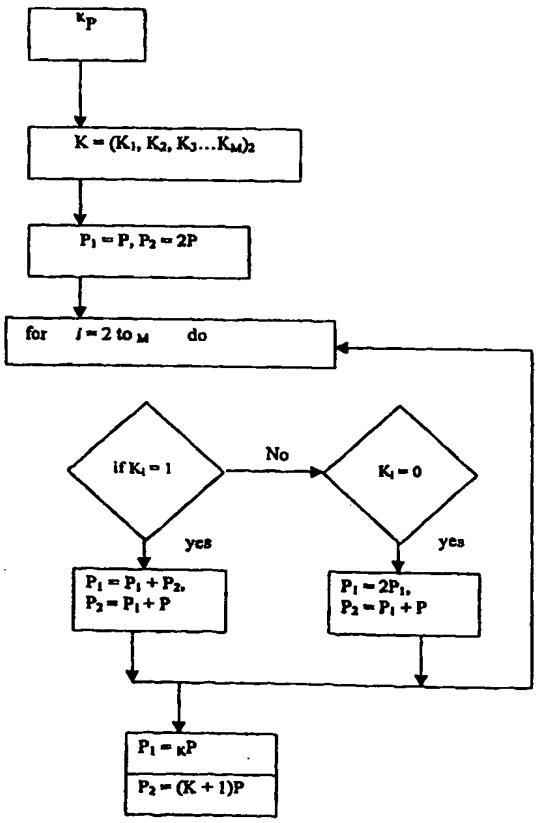
(81) Designated States: AL, AM, AT, AU, AZ, BA, BB, BG, BR, BY, CA, CH, CN, CU, CZ, DE, DK, EE, ES, FI, GB, GD, GE, GH, GM, HR, HU, ID, IL, IS, JP, KE, KG, KP, KR, KZ, LC, LK, LR, LS, LT, LU, LV, MD, MG, MK, MN, MW, MX, NO, NZ, PL, PT, RO, RU, SD, SE, SG, SI, SK, SL, TJ, TM, TR, TT, UA, UG, US, UZ, VN, YU, ZW, ARIPO patent (GH, GM, KE, LS, MW, SD, SL, SZ, UG, ZW), Eurasian patent (AM, AZ, BY, KG, KZ, MD, RU, TJ, TM), European patent (AT, BE, CH, CY, DE, DK, ES, FI, FR, GB, GR, IE, IT, LU, MC, NL, PT, SE), OAPI patent (BF, BJ, CF, CG, CI, CM, GA, GN, GW, ML, MR, NE, SN, TD, TG).

**Published**
*With international search report.*

(54) Title: ACCELERATED FINITE FIELD OPERATIONS ON AN ELLIPTIC CURVE

(57) Abstract

A method for multiplication of a point P on elliptic curve E by a value k in order to derive a point kP comprises the steps of representing the number k as vector of binary digits stored in a register and forming a sequence of point pairs (P1, P2) wherein the point pairs differed most by P and wherein the successive series of point pairs are selected either by computing (2mP,(2m+1)P) from (mP,(m+1)P) or ((2m+1)P,(2m+2)P) from (mP,(m+1)P). The computations may be performed without using the y–coordinate of the points during the computation while allowing the y–coordinate to be extracted at the end of the computations, thus, avoiding the use of inversion operations during the computation and therefore, speeding up the cryptographic processor functions. A method is also disclosed for accelerating signature verification between two parties.



*FOR THE PURPOSES OF INFORMATION ONLY*

Codes used to identify States party to the PCT on the front pages of pamphlets publishing international applications under the PCT.

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| **AL** | Albania | **ES** | Spain | **LS** | Lesotho | **SI** | Slovenia |
| **AM** | Armenia | **FI** | Finland | **LT** | Lithuania | **SK** | Slovakia |
| **AT** | Austria | **FR** | France | **LU** | Luxembourg | **SN** | Senegal |
| **AU** | Australia | **GA** | Gabon | **LV** | Latvia | **SZ** | Swaziland |
| **AZ** | Azerbaijan | **GB** | United Kingdom | **MC** | Monaco | **TD** | Chad |
| **BA** | Bosnia and Herzegovina | **GE** | Georgia | **MD** | Republic of Moldova | **TG** | Togo |
| **BB** | Barbados | **GH** | Ghana | **MG** | Madagascar | **TJ** | Tajikistan |
| **BE** | Belgium | **GN** | Guinea | **MK** | The former Yugoslav Republic of Macedonia | **TM** | Turkmenistan |
| **BF** | Burkina Faso | **GR** | Greece | | | **TR** | Turkey |
| **BG** | Bulgaria | **HU** | Hungary | **ML** | Mali | **TT** | Trinidad and Tobago |
| **BJ** | Benin | **IE** | Ireland | **MN** | Mongolia | **UA** | Ukraine |
| **BR** | Brazil | **IL** | Israel | **MR** | Mauritania | **UG** | Uganda |
| **BY** | Belarus | **IS** | Iceland | **MW** | Malawi | **US** | United States of America |
| **CA** | Canada | **IT** | Italy | **MX** | Mexico | **UZ** | Uzbekistan |
| **CF** | Central African Republic | **JP** | Japan | **NE** | Niger | **VN** | Viet Nam |
| **CG** | Congo | **KE** | Kenya | **NL** | Netherlands | **YU** | Yugoslavia |
| **CH** | Switzerland | **KG** | Kyrgyzstan | **NO** | Norway | **ZW** | Zimbabwe |
| **CI** | Côte d'Ivoire | **KP** | Democratic People's Republic of Korea | **NZ** | New Zealand | | |
| **CM** | Cameroon | | | **PL** | Poland | | |
| **CN** | China | **KR** | Republic of Korea | **PT** | Portugal | | |
| **CU** | Cuba | **KZ** | Kazakstan | **RO** | Romania | | |
| **CZ** | Czech Republic | **LC** | Saint Lucia | **RU** | Russian Federation | | |
| **DE** | Germany | **LI** | Liechtenstein | **SD** | Sudan | | |
| **DK** | Denmark | **LK** | Sri Lanka | **SE** | Sweden | | |
| **EE** | Estonia | **LR** | Liberia | **SG** | Singapore | | |

# ACCELERATED FINITE FIELD OPERATIONS ON AN ELLIPTIC CURVE

This invention relates to a method of accelerating operations in a finite field, and in particular, to operations performed in a field $F_{2^m}$ such as used in encryption systems.

## BACKGROUND OF THE INVENTION

Finite fields of characteristic two in $F_{2^m}$ are of interest since they allow for the efficient implementation of elliptic curve arithmetic. The field $F_{2^m}$ can be viewed as a vector space of dimension $m$ over $F_2$. Once a basis of $F_{2^m}$ over $F_2$ has been chosen the elements of $F_{2^m}$ can be conveniently represented as vectors of elements zero or one and of length $m$. In hardware, a field element is stored in a shift register of length $m$. Addition of field elements is performed by bitwise XOR-ing ($\oplus$) the vector representations and takes one clock cycle.

Digital signatures are used to confirm that a particular party has sent a message and that the contents have not been altered during transmission.

A widely used set of signature protocols utilizes the ElGamal public key signature scheme that signs a message with the sender's private key. The recipient may then verify the signature with the sender's public key.

Various protocols exist for implementing such a scheme and some have been widely used. In each case however the recipient is required to perform a computation to verify the signature. Where the recipient has adequate computing power this does not present a particular problem but where the recipient has limited computing power, such as in a "Smart card " application, the computations may introduce delays in the verification process.

Public key schemes may be implemented using one of a number of groups in which the discrete log problem appears intractable but a particularly robust implementation is that utilizing the characteristics of points on an elliptic curve over a finite field. This implementation has the advantage that the requisite security can be obtained with relatively small orders of field compared with for example with implementations in $Z_p^*$ and therefore reduces the bandwidth required for communicating the signatures.

In a typical implementation a signature component s has the form:

$s = ae + k \pmod n$ where:

P is a point on the curve, which is a predefined parameter of the system;

k is a random integer selected as a short term private or session key, and has a corresponding short term public key R = kP;

a is the long term private key of the sender and has a corresponding public key aP = Q;

e is a secure hash, such as the SHA hash function, of a message m and short term public key R; and

n is the order of the curve.

The sender sends to the recipient a message including m, s, and R and the signature is verified by computing the value $R'$ = (sP-eQ) which should correspond to R. If the computed values are equivalent then the signature is verified.

In order to perform the verification it is necessary to compute a number of point multiplications to obtain sP and eQ, each of which is computationally complex.

If $F_q$ is a finite field, then elliptic curves over $F_q$ can be divided into two classes, namely supersingular and non-supersingular curves. If $F_q$ is of characteristic 2, i.e. $q = 2^M$, then the classes are defined as follows.

i) The set of all solutions to the equation $y^2 + ay = x^3 + bx + c$ where $a,b,c \in F_q, a \neq 0$, together with a special point called the point at infinity O is a supersingular curve over $F_q$.

ii) The set of all solutions to the equation $y^2 + xy = x^3 + ax^2 + b$ where $a,b \in F_q, b \neq 0$, together with a special point called the point at infinity O is a nonsupersingular curve over $F_q$.

By defining an appropriate addition on these points, we obtain an additive abelian group. The addition of two points $P(x_1, y_1)$ and $Q(x_2, y_2)$ for the supersingular elliptic curve $E$ with $y^2 + ay = x^3 + bx + c$ is given by the following:-

If $P = (x_1, y_1) \in E$; then define $-P = (x_1, y_1 + a)$, $P + O = O + P = P$ for all $P \in E$.

If $Q = (x_2, y_2) \in E$ and $Q \neq -P$, then the point representing the sum of $P + Q$, is denoted $(x_3, y_3)$, where

$$x_3 = \left\{ \left( \frac{y_1 \oplus y_2}{x_1 \oplus x_2} \right)^2 \oplus x_1 \oplus x_2 \quad (P \neq Q) \right.$$

or

$$x_3 = \left\{ \frac{x_1^4 \oplus b^2}{a^2} \quad (P = Q) \right.$$

and

$$y_3 = \left\{ \left( \frac{y_1 \oplus y_2}{x_1 \oplus x_2} \right) \oplus (x_1 \oplus x_3) \oplus y_1 \oplus a \quad (P \neq Q) \right.$$

or

$$y_3 = \left\{ \left( \frac{x_1^2 \oplus b}{a} \right) \oplus (x_1 \oplus x_3) \oplus y_1 \oplus a \quad (P = Q) \right.$$

The addition of two points $P(x_1, y_1)$ and $Q(x_2, y_2)$ for the nonsupersingular elliptic curve $y^2 + xy = x^3 + ax^2 + b$ is given by the following:-

If $P = (x_1, y_1) \in E$ then define - $P = (x_1, y_1 + x_1)$. For all $P \in E$, $O + P = P + O = P$. If $Q = (x_2, y_2) \in E$ and $Q \neq -P$, then $P + Q$ is a point $(x_3, y_3)$, where

$$x_3 = \left\{ \left( \frac{y_1 \oplus y_2}{x_1 \oplus x_2} \right)^2 \oplus \frac{y_1 \oplus y_2}{x_1 \oplus x_2} \oplus x_1 \oplus x_2 \oplus a \quad (P \neq Q) \right.$$

or

$$x_3 = \left\{ x_1^2 \oplus \frac{b}{x_1^2} \quad (P = Q) \right.$$

]and

$$y_3 = \left\{ \left( \frac{y_1 \oplus y_2}{x_1 \oplus x_2} \right) \oplus (x_1 \oplus x_3) \oplus x_3 \oplus y_1 \quad (P \neq Q) \right.$$

or

$$y_3 = \begin{cases} x_1^2 \oplus \left( x_1 \oplus \frac{y_1}{x_1} \right) \oplus x_3 \oplus x_3 & (P = Q) \end{cases}$$

Now supersingular curves are preferred, as they are more resistant to the MOV attack. It can be seen that computing the sum of two points on $E$ requires several multiplications, additions, and inverses in the underlying field $F_{2^m}$. In turn, each of these operations requires a sequence of elementary bit operations.

When implementing cryptographic operations in ElGamal or Diffie-Hellman schemes or generally most cryptographic operations with elliptic curves, one is required to compute $kP = P + P + ... + P$ ($P$ added $k$ times) where $k$ is a positive integer and $P \in E$. This requires the computation of $(x_3,y_3)$ to be computed k-1 times. For large values of k which are typically necessary in cryptographic applications, this has previously been considered impractical for data communication. If $k$ is large, for example 1024 bits, $kP$ would be calculated by performing $2^{1024}$ additions of $P$.

Furthermore, in a multiplicative group, multiplications and inversions are extremely computationally intensive, with field inversions being more expensive than field multiplications. The inversion operation needed when adding two points can be eliminated by resorting to projective coordinates. The formula for addition of two points however, requires a larger number of multiplications than is required when using affine coordinates.

In a paper entitled "Elliptic Curve Cryptosystems and Their Implementation" by Vanstone et al., published in The Journal of Cryptology, a method is described for adding two points by converting to projective coordinates and thus eliminating the inversion computation. However, the overall gain in speed by elimination of the inversion is at the expense of space. Extra registers are required to store P and Q and also to store intermediate results when doing the addition. Furthermore, this method requires the use of the y-coordinate in the calculation.

## SUMMARY OF THE INVENTION

It is therefore an object of the present invention to provide a method and apparatus in which some of the above disadvantages are obviated or mitigated.

It is a further object of the invention to provide a method of multiplying finite field elements, and which may be implemented relatively efficiently on a processor with limited processing capability, such as a smart card or the like.

It is a still further object of the present invention to provide a method and apparatus in which signature verification may be accelerated in elliptic curve encryption systems.

In accordance with this invention there is provided a method of determining a multiple of a point $P$ on an elliptic curve defined over a field $F_{2^M}$, said method comprising steps of;

a) representing the number $k$ as a vector of binary digits $k_i$;

b) forming a pair of points $P_1$ and $P_2$, wherein the point $P_1$ and $P_2$ differ at most by $P$; and

c) selecting each of the $k_i$ in turn and for each of the $k_i$,

upon the $k_i$ being a one, adding the pair of points $P_1$ and $P_2$ to form a new point $P_1$ and adding the point $P$ to $P_1$ to form a new point $P_2$, the new points replacing the pair of points $P_1$ and $P_2$; or

upon the $k_i$ being a zero, doubling the point $P_1$ to form a new point $P_1$ and adding the point $P$ to form a new point $P_2$, the new points replacing the pair of points $P_1$ and $P_2$, whereby the product $kP$ is obtained from the point $P_1$ in $M$-1 steps and wherein $M$ represents the number of digits in $k$.

Furthermore, the inventors have implemented a method whereby computation of a product kP can be performed without the use of the y coordinate of the point P during computation.

## BRIEF DESCRIPTION OF THE DRAWINGS

Embodiments of the present invention will now be described by way of example only with reference to the accompanying drawings in which: -

Figure 1 is a schematic representation of a data communication system;

Figure 2 is a schematic diagram of an encryption/decryption unit;

Figure 3 is a flow chart for computing a multiple of a point;

Figure 4 is a flow chart showing the extraction of an y-coordinate;

## DETAILED DESCRIPTION OF A PREFERRED EMBODIMENT

Referring to Figure 1, a data communication system 2 includes a pair of correspondents, designated as a sender 10, and a recipient 12, connected via a communication channel 14. Each of the correspondents 10, 12 includes an encryption/decryption unit 16 associated therewith that may process digital information and prepare it for transmission through the channel 14 as will be described below. The encryption/decryption units implement amongst, others key exchange protocols and an encryption/decryption algorithm.

The module 16 is shown schematically in Figure 2 and includes an arithmetic logic unit 20 to perform the computations including key exchange and generation. A private key register 22 contains a private key, d, generated for example as a 155 bit data string from a random number generator 24, and used to generate a public key stored in a public key register 26. A base point register 28 contains the co-ordinates of a base point P that lies in the elliptic curve selected with each co-ordinate (x, y), represented as a 155 bit data string. Each of the data strings is a vector of binary digits with each digit being the coefficient of an element of the finite field in the normal basis representation of the co-ordinate.

The elliptic curve selected will have the general form $y^2 + xy = x^3 + ax^2 + b$ and the parameters of that curve, namely the coefficients a and b are stored in a parameter register 30. The contents of registers 22, 24, 26, 28, 30 may be transferred to the arithmetic unit 20 under control of a CPU 32 as required.

The contents of the public key register 26 are also available to the communication channel 14 upon a suitable request being received. In the simplest implementation, each encryption module 16 in a common secure zone will operate with the same curve and base point so that the contents of registers 28 and 30 need not be accessible. If further sophistication is required, however, each module 16 may select its own curve and base point in which case the contents of registers 28, 30 have to be accessible to the channel 14.

The module 16 also contains an integer register 34 that receives an integer k, the session seed, from the generator 24 for use in encryption and key exchange. The module 16 has a random access memory (RAM) 36 that is used as a temporary store as required during computations.

In accordance with a general embodiment, the sender assembles a data string, which includes amongst others, the public key Q of the sender, a message m, the senders

short term public key R and a signature component s of the sender. When assembled the data string is sent over the channel 4 to the intended recipient 12.

For simplicity it will be assumed that the signature component s of the sender 12 is of the form $s = ae + k \pmod n$ as discussed above although it will be understood that other signature protocols may be used. To verify the signature sP-eQ must be computed and compared with R.

Thus a first step of the recipient is to retrieve the value of Q from the string. A hash value e may also be computed from the message m and the coordinates of the point R. The recipient is then able to perform the verification by computing sP and eQ.

In order to accelerate the calculation of sP or eQ the recipient may adopt the following to calculate the coordinates of the new point sP, in order to avoid performing the several multiplications, additions and inverses in the underlying field $F_2^m$. The recipient may calculate sP by resorting to the expedient of a "double and add" method as shown in figure 3.

Referring to figure 3 one embodiment of the invention illustrating a "double and add" method for multiplication a point $P$ on an elliptic curve $E$ by a value $k$ in order to derive a point $kP$ is implemented by initially representing $k$ in its binary form. Next a successive series of point pairs ($mP$, $(m+1)P$) are set up. Each successive digit of $k$ is considered in turn, upon the occurrence of a zero value digit in the binary representation of $k$, the first of the pair of points is doubled and one is added to the second of the pair of points i.e compute (2mP,(2m+1)P) from (mP,(m+1)P). Alternatively upon the occurrence of a one value in the binary representation of $k$, the first of the pair is formed from the sum of the previous pair of points and the second of the pair is formed by adding one to the first of the pair i.e. compute ((2m+1)P,(2m+2)P) from (mP,(m+1)P).

This is illustrated in the following short example: in which $k = 23$. The value of $k$ may be represented in binary as pairs (11011). Applying the above rule to a pair of points ($P$, $2P$) we get the successive sequence of point, ($2P$, $3P$); ($5P$, $6P$); ($11P$, $12P$); and finally ($23P$, $24P$). The first of the pair is thus the required point.

Thus, it may be seen the final result $23P$ is obtained by performing a series of "double and add" operations on a pair of points in the field wherein the pair of points in a given pair differ by P. Furthermore the number of "double and add" operations equals at most one less than the number of bits in $k$ i.e. $(m - 1)$ times. This method of "double and add" has a distinct advantage for large values of $k$ in reducing the number of operations to

be performed by a processor. This may be contrasted with performing $k$ double and adds on a single point P as described earlier in the background of the invention.

Turning back to the calculation of sP and eQ, the recipient may thus apply the above embodiment to calculating sP for the nonsupersingular elliptic curve $y^2 + xy = x^3 + ax^2 + b$, E defined over $F_{2^m}$.

If $P_1 = (x_1, y_1)$ and $P_2 = (x_2, y_2)$, $P_1 \neq \pm P_2$, are points on the curve E then we can define $P_1 + P_2 = (x_3, y_3)$ where,

$$x_3 = \lambda^2 + \lambda + x_1 + x_2 + a \quad (1)$$

wherein the slope of the curve is given by:

$$\lambda = \frac{y_2 + y_1}{x_2 + x_1}$$

Similarly, if $-P_2 = (x_2, y_2+x_2)$ and $P_1 - P_2 = (x_4, y_4)$ then,

$$x_4 = \overline{\lambda}^2 + \overline{\lambda} + x_1 + x_2 + a = \lambda^2 + \frac{x}{(x_1 + x_2)^2} + \lambda + \frac{x_2}{x_1 + x_2} + x_1 + x_2 + a \quad (2)$$

where

$$\overline{\lambda} = \frac{y_2 + x_2 + y_1}{x_2 + x_1} = \frac{x_2}{x_2 + x_1} + \lambda$$

if we add $x_3$ and $x_4$ then,

$$x_3 + x_4 = \frac{x}{(x_1 + x_2)^2} + \frac{x_2}{x_1 + x_2} = \frac{x_1 x_2}{(x_1 + x_2)^2} \quad (3)$$

To compute the x-coordinate $x_3$ of $(P_1 + P_2)$ we only need the x-coordinates of $P_1$, $P_2$ and $(P_1 - P_2)$, however the computation is not optimally efficient as it requires inversions. It may also be noted that the y-coordinate is not needed in these calculations.

Referring back to figure 2, the value $kP$ may be calculated using the "double and add" method. Whenever a new pair of points is computed the addition formula of equation (3) above is used and this is done $m$ times.

Thus we have a formula for $x_3$ involving $x_1$, $x_2$ and $x_4$. Unfortunately, this formula includes an inversion, which is costly. We can modify this equation as follows, suppose the values of $x_1$, $x_2$ and $x_3$ are given by $\frac{x_1}{z_1}$, $\frac{x_2}{z_2}$, $\frac{x_3}{z_3}$, where of $x_1$, $x_2$, $x_3$ $z_1$, $z_2$, $z_3$ are values

maintained during the double and add algorithm. Then substituting these new representations into formula (3), we find

$$\frac{x_3}{z_3} = x_4 + \frac{\frac{x_1 x_2}{z_1 z_2}}{(\frac{x_1}{z_1} + \frac{x_2}{z_2})^2} = x_4 + \frac{x_1 x_2 z_1 z_2}{(x_1 z_2 + x_2 z_1)^2} = \frac{x_4 (x_1 z_2 + x_2 z_1)^2 + x_1 x_2 z_1 z_2}{(x_1 z_2 + x_2 z_1)^2}$$

Therefore, if we take $x_3 = x_4 (x_1 z_2 + x_2 z_1)^2 + x_1 x_2 z_1 z_2$, and $z_3 = (x_1 z_2 + x_2 z_1)^2$. We can execute the "double & add" algorithm of figure 3 (using this new representation) and avoid the computation of an inversion for most of the algorithm.

From equations for $x_3$ and $z_3$ above it may be seen that $x_3$ may be calculated by performing at most four multiplication operations.

The sum of the points $P_1$ and $P_2$ are expressed in terms of $x_3$ and $z_3$ is obtained without having to perform a relatively costly inversion on the x-coordinate, and can be computed using at most four multiplies and two squares. The remaining operations of addition and squaring are relatively inexpensive with regard to computational power. The computation of the term $(x_1 z_2 + x_2 z_1)^2$ is obtained by a cyclic shift of the normal basis representation of the value within parentheses for which a general-purpose processor can perform relatively easily. At the end of the algorithm we can convert back to our original representation if required.

Referring back to figure 3, now in order to double point P $(x_1, y_1)$, let $2(x_1, y_1) = (x_3, y_3)$ then as before if the equation of the elliptic curve E is given by $y^2 + xy = x^3 + ax^2 + b$ over $F_2^m$, the x-coordinate of the point 2P is represented as

$$x_3 = x_1^2 + \frac{b}{x_1^2}.$$

Once again representing the coordinates in terms of the projective coordinates we obtain

$$x_3 = x_1^4 + b z_1^4$$

and

$$z_3 = (x_1 z_1)^2$$

or

$$x_3 = (x_1 + \sqrt[4]{b} z_1)$$

By making b relatively small the computationally expensive operations may be reduced to approximately one multiplication operation for the $z_3$ term. We can

precompute $\sqrt[4]{b}$ and calculate $x_3$ according to the last equation, thus requiring two less squares. Alternatively, as mentioned earlier in a normal basis representation the computation of $x_1^4$ and $z_1^4$ is obtained by two cyclic shifts of the representation of the respective values, while $(x_1 z_1)^2$ is obtained by a single cyclic shift of the product.

Applying the earlier outlined "double and add" method of figure 3, we observe that for a scalar k of m bits and calculation of kP defined over $F_2^m$ requires at most (m-1) double and add operations. From the above discussion a double operation on points of an elliptic curve are achieved by performing at most two multiplication operations, while the add operation is achieved by performing at most four multiplication operations. Thus to compute the x-coordinate of kP using the method of this invention would require at most six times (m-1) multiplication operations.

Once the x values have been calculated, as above, y-coordinate values may also be determined. However, for each x-coordinate there exists at most two y-coordinates. For example, in the final step of obtaining a point 24P, both points 23P and P would be known, since 24P may be expressed as 23P + P = 24P. Assume the x-coordinate $x_{23}$ of the point A = 23P have been obtained as described earlier. Then, by substituting $x_{23}$ into the elliptic curve equation E and solving the resulting quadratic equation, two values of y are obtained corresponding to points $A = (x_{23}, y_{23}^{(1)})$ and $B = (x_{23}, y_{23}^{(2)})$. Next, by substitution, the x-coordinate $x_{24}$ obtained through calculating 24P = P + 23P into the elliptic curve equation will produce two points $(x_{24}, y_{24}^{(1)})$ and $(x_{24}, y_{24}^{(2)})$. The two points thus obtained are stored. To the point A + B are added, point P using ordinary point addition to produce corresponding points $A + P = (x_a, y_a)$ and $B + P = (x_b, y_b)$, respectively. Point $(x_a, y_a)$ is compared to points $(x_{24}, y_{24}^{(1)})$ and $(x_{24}, y_{24}^{(2)})$, respectively. If none of the points match, then $(x_b, y_b)$ is the correct point, otherwise $(x_a, y_a)$ is the correct point. Thus, it may be seen that multiples of a point P may be easily calculated without knowing the y-coordinate and, furthermore, the y-coordinate may be obtained at the end of the calculation, if so desired.

Thus, for example referring back to the ElGamal scheme for elliptic curves one is required to compute $r = kP = (x,y)$. In this case one can drop the y-coordinate and produce a hash of a message m and the x-coordinate e = h(m//x). The sender then sends to a recipient a message including a signature s and the hash e. The signature s has the form s = (de + k) mod n, where d is the private key of the sender and k is a random number generated by the sender. The recipient then verifies the signature by calculating sP -eQ =

r. Both sP and eQ may be calculated by utilizing the "double and add" method of this invention. The x values of sP and eQ each produce two possible values of y: $(x_1, y_1^{(1)})$, $(x_1, y_1^{(2)})$ and $(x_2, y_2^{(1)})$, $(x_2, y_2^{(2)})$ when substituted back into the elliptic curve equation E. When the point subtraction is performed between permutations of these points, the correct y will thus produce the appropriate matching r. If none of these substitutions produce a matching r, then the signature is not verified.

Referring to figure 4, a schematic diagram of a further method for determining the y-coordinate of $k$P derived according to the method described with respect to figure 3, and given the point P = (x, y) and the x-coordinate $\bar{x}$ of $(k-1)$P and x′ of $kP$ is shown generally by numeral 50. As may be noted with respect to figure 3 in computing the x-coordinate of $kP$ the x-coordinate of $(k-1)$P is also calculated.

Thus, initially substitute into the elliptic curve equation to obtain a value of y′ such that the point (x′,y′) is on the curve. Next at step 54 assign the point Q to (x′,y′). Next complete a point Q-P = (x″,y″) by simple point subtraction 55. The derived x-coordinate x" is compared to the x-coordinate $\bar{x}$ of (k-1) at step 56 and if $x'' = \bar{x}$, then y′ is the y-coordinate of $k$P, otherwise y′ is the y-coordinate of -$k$P. It may be noted that this method works if 0 < k < order of point P.

Utilizing the method of the subject invention to compute $k$P it is also possible to compute $(k+1)$P such that the x-coordinates on $k$P and $(k+1)$P are available. In this case the y-coordinate may be derived by computing Q+P = (x″, y″) and comparing the coordinate x″ to the x-coordinate of $(k+1)$P.

Referring to figure 5, a further application of an embodiment of the invention to verification of elliptic curve signatures is indicated generally by numeral 70. Once again it is assumed that the first correspondent 10 includes a private key random integer $d$ and a corresponding public key $Q$ derived from computing the point $Q = dP$. In order to sign a message $M$, a hash value $e$ is computed from the message $M$ using a hash function $H$. Next, a random integer $k$ is selected as a private session key. A corresponding public session key $kP$ is calculated from the random integer $k$. The first correspondent then represents the $x$-coordinate of the point $kP$ as an integer $z$ and then calculates a first signature component $r = z \bmod n$.

Next, a second signature component $s = k^{-1}(e + dr) \bmod n$ is also calculated. The signature components $s$ and $r$ and a message $M$ is then transmitted to the second correspondent 12. In order for the second correspondent 12 to verify the signature $(r,s)$ on

$M$, the second correspondent looks up the public key $Q$ of the first correspondent 10. A hash $e'$ of the message $M$ is calculated using the hash function $H$ such that $e' = H(M)$. A value $c = s^{-1} \bmod n$ is also calculated. Next, integer values $u_1$ and $u_2$ are calculated such that $u_1 = e'c \bmod n$ and u2 = $rc \bmod n$. In order that the signature be verified, the value $u_1P + u_2Q$ must be calculated. Since $P$ is known and is a system wide parameter, the value $u_1P$ may be computed quickly using pre-computed multiple of P. For example, these values may be combined from a pre-stored table of doubles of $P$, i.e. $2P$, $4P$, $8P$, etc. On the other hand however, the point $Q$ is current and varies from user to user and, therefore, the value $u_2Q$ may take some time to compute and generally cannot be pre-computed.

However, by resorting to the expedient of the method disclosed in the subject invention, verification of the signature may be significantly accelerated. Normally, the point $R = u_1P + u_2Q$ is computed. The field element x of the point R = (x,y) is converted to an integer $z$, and a value $v = z \bmod n$ is computed. If v = r, then the signature is valid.

Alternatively, a technique which takes advantage of "double & add" to compute $u_2Q$ if the modular inverse of $u_2$ is calculated $u_2^* = u_2^{-1} \bmod n$, then R can be expressed as $u_2(u_1 u_2^* P + Q)$, i.e. making use of the identity $u_2 u_2^* = 1$. The value $u_1 u_2^*$ is an integer and, therefore, may be easily computed. Thus, the point $u_1 u_2^* P$ may be easily calculated or assembled from the previously stored values of multiples of $P$. The point $Q$ is then added to the point $u_1 u_2^* P$, which is a single addition, to obtain a new point $R'$.

Thus, in order to verify the signatures, the recipient need only to determine the x coordinate of the value $u_2R'$. This calculation may be performed using the "double and add" method as described with reference to figure 3. If this is equal to r, then the signature is verified. The resulting value is the $x$-coordinate of the point $u_1P + u_2Q$. The value v = x mod n is computed and verified against r. It may be noted that in this scheme, the y-coordinate is not used in signature generation or verification and, hence, computing is not mandatory. However, alternative schemes for both x and y-coordinates may be utilized in these cases and the y coordinate may be derived as described earlier or the two y-coordinates corresponding to the given x-coordinate may be calculated and each used to attempt to verify the signature. Should neither satisfy this comparison, then the signature is invalid. That is, since verification requires computing the point $R = U_1P + U_2Q$. This can be done as follows. Transmit only the x coordinate of Q, compute the x-coordinate of $U_2Q$,, by using either the "double & add" of figure 3 or on $E(F_p)$. Try both points corresponding to this x-coordinate to see if either verifies.

Referring back to figure 1 if keys are transferred between the correspondents of the form kP then to reduce the bandwidth it is possible for the sender to transmit only one of the co-ordinates of kP and compute the other co-ordinate at the receiver. For example if the field elements are 155 bits for $F_2^{155}$, an identifier, for example a single bit of the correct value of the other co-ordinate, may also be transmitted. This permits the possibilities for the second co-ordinate to be computed by the recipient and the correct one identified from the identifier.

Referring therefore to Figure 1, the transmitter 10 initially retrieves as the public key dP of the receiver 12, a bit string representing the co-ordinate $x_0$ and a single bit of the co-ordinate $y_0$.

The transmitter 10 has the parameters of the curve in register 30 and therefore may use the co-ordinate $x_0$ and the curve parameters to obtain possible values of the other co-ordinate $y_0$ from the arithmetic unit 20.

For a curve of the form $y^2 + xy = x^3 + ax^2 + b$ and a co-ordinate $x_0$, then the possible values $y_1, y_2$ for $y_0$ are the roots of the quadratic $y^2 + x_0y = x_0^3 + ax_0^2 + b$.

By solving for y, in the arithmetic unit 20 two possible roots will be obtained and comparison with the transmitted bit of information will indicate which of the values is the appropriate value of y.

The two possible values of the second co-ordinate ($y_0$) differ by $x_0$, i.e. $y_1 = y_2 + x_0$. Since the two values of $y_0$ differ by $x_0$, then $y_1$ and $y_2$ will always differ where a "1" occurs in the representation of $x_0$. Accordingly the additional bit transmitted is selected from one of those positions and examination of the corresponding bit of values of $y_0$, will indicate which of the two roots is the appropriate value.

The receiver 10 thus can generate the co-ordinates of the public key dP even though only 156 bits are retrieved.

Similar efficiencies may be realized in transmitting the session key kP to the receiver 12 as the transmitter 10 need only forward one co-ordinate, $x_0$ and the selected identifying bit of $y_0$. The receiver 12 may then reconstruct the possible values of $y_0$ and select the appropriate one.

In the field $F_{2^m}$ it is not possible to solve for y using the quadratic formula as 2a = 0. Accordingly, other techniques need to be utilised and the arithmetic unit 20 is particularly adapted to perform this efficiently.

In general provided $x_0$ is not zero, if $y = x_0 z$ then $x_0^2 z^2 + x_0^2 z = x^3_0 + a x_0^2 + b$. This may be written as $z^2 + z = x_0 + a + \frac{b}{x_0^2} 1 = c$.

i.e. $z^2 + z = c$.

If m is odd then <u>either</u> $z = c + c^4 + c^{16} + \ldots + c^{2m-1}$

<u>or</u> $z = 1 + c + \ldots.. + c^{2m-1}$ to provide two possible values for $y_0$.

A similar solution exists for the case where m is even that also utilises terms of the form $c^{2^g}$.

This is particularly suitable for use with a normal basis representation in $F_{2^m}$.

As noted above, raising a field element in $F_{2^m}$ to a power g can be achieved by a g fold cyclic shift where the field element is represented as a normal basis.

Accordingly, each value of z can be computed by shifting and adding and the values of $y_0$ obtained. The correct one of the values is determined by the additional bit transmitted.

The use of a normal basis representation in $F_{2^m}$ therefore simplifies the protocol used to recover the co-ordinate $y_0$.

If $P = (x_0\ y_0)$ is a point on the elliptic curve $E : y^2 + xy = x^3 + ax^2 + b$ defined over a field $F_{2^m}$, then $y_0$ is defined to be 0 if $x_0 = 0$; if $x_0 \neq 0$ then $y_0$ is defined to be the least significant bit of the field element $y_0 \cdot x_0^{-1}$.

The x-coordinate $x_0$ of P and the bit $y_0$ are transmitted between the transmitter 10 and receiver 12. Then the y-coordinate $y_0$ can be recovered as follows.

1. If $x_0 = 0$ then $y_0$ is obtained by cyclically shifting the vector representation of the field element b that is stored in parameter register 30 one position to the left. That is, if $b = b_{m-1} b_{m-2} \ldots b_1 b_0$ then $y = b_{m-2} \ldots b_1 b_0 b_{m-1}$

2. If $x_0 \neq 0$ then do the following:

   2.1 Compute the field element $c = x_0 + a + b x_0^{-2}$ in $F_2^m$.

   2.2 Let the vector representation of c be

   $c = c_{m-1}\ c_{m-2} \ldots c_1 c_0$.

   2.3 Construct a field element $z = z_{m-1} z_{m-2} \ldots z_1 z_0$ by setting

   $z_0 = y_0$,

   $z_1 = c_0 \oplus z_0$,

$$z_2 = c_1 \oplus z_1,$$

$$z_{m-2} = c_{m-3} \oplus z_{m-3},$$

$$z_{m-1} = c_{m-2} \oplus z_{m-2}.$$

2.4 Finally, compute $y_0 = x_0 \cdot z$.

It will be noted that the computation of $x_0^{-2}$ can be readily computed in the arithmetic unit 20 as described above and that the computation of $y_0$ can be obtained from the multiplier 48.

In the above examples, the identification of the appropriate value of $y_0$ has been obtained by transmission of a single bit and a comparison of the values of the roots obtained. However, other indicators may be used to identify the appropriate one of the values and the operation is not restricted to encryption with elliptic curves in the field $GF(2^m)$. For example, if the field is selected as $Z_p$ $p = 3 (\text{mod } 4)$ then the Legendre symbol associated with the appropriate value could be transmitted to designate the appropriate value. Alternatively, the set of elements in Zp could be subdivided into a pair of subsets with the property that if y is in one subset, then -y is in the other, provided $y \neq 0$. An arbitrary value can then be assigned to respective subsets and transmitted with the co-ordinate $x_0$ to indicate in which subset the appropriate value of $y_0$ is located. Accordingly, the appropriate value of $y_0$ can be determined. Conveniently, it is possible to take an appropriate representation in which the subsets are arranged as intervals to facilitate the identification of the appropriate value of $y_0$. It may be noted that one of the methods described earlier may also be sued to derive the coordinate.

These techniques are particularly suitable for encryption utilizing elliptic curves but may also be used with any algebraic curves and have applications in other fields such as error correcting coding where co-ordinates of points on curves have to be transferred.

It will be seen therefore that by utilising an elliptic curve lying in the finite field $GF_2^m$ and utilising a normal basis representation, the computations necessary for encryption with elliptic curves may be efficiently performed. Such operations may be implemented in either software or hardware and the structuring of the computations makes the use of a finite field multiplier implemented in hardware particularly efficient.

In a further embodiment of the invention for improving the efficiency of computing scalar multiples of an elliptic curve point in $F_p$ or $F_2m$ is described below. Consider the generalized elliptic curve equation $E: y^2 + xy = x^3 + ax^2 + b, b \neq 0$ over $F_2m$.

If $P=(x_1,y_1)$ is a generator, then using the method as described with respect to figure 3 and projective coordinates, we get $kP$, $(k+1)P$ where $k$ is the scalar and $kP=(X_2,Z_2)$, $(k+1)P=(X_3,Z_3)$.

Our objective is to determine the affine coordinates of $kP=(x_2,y_2)$. We know, of course, that $x_2 = \frac{X_2}{Z_2}$

Suppose the affine coordinates of $(k+1)P=(x_3,y_3)$.

Then, $x_3 = \lambda^2 + \lambda + x_1 + x_2 + a, \lambda = \frac{y_1 + y_2}{x_1 + x_2}$

$$\lambda^2 + \lambda + (x_1 + x_2) + a = \frac{y_1^2 + y_2^2 + (x_1 + x_2)(y_1 + y_2) + (x_1 + x_2)^3 + a(x_1 + x_2)^2}{(x_1 + x_2)^2}$$

$$= \{y_2^2 + x_1 y_2 + x_2 y_1 + x_2 y_2 + x_1^2 x_2 + x_1 x_2^2 + x_2^3 + a x_2^2\} \frac{1}{(x_1 + x_2)^2}$$

$$= \frac{\{b + b + x_1 y_2 + x_2 y_1 + x_1^2 x_2 + x_1 x_2^2\}}{(x_1 + x_2)^2}$$

$$= \frac{x_1 y_2 + x_2 y_1 + x_2^2 x_1 + x_1^2 x_2}{(x_1 + x_2)^2}$$

$$= \frac{x_1 y_2 + x_2 y_1 + x_1 x_2 (x_1 + x_2)}{(x_1 + x_2)^2}$$

Solving for $y_2$ we get

$$y_2 = \frac{1}{x_1}\{x_3 (x_1 + x_2)^2 + x_2 y_1 + x_1 x_2 (x_1 + x_2)\} \qquad (*)$$

We know $x_1$ and $y_1$ but must compute

$$x_2 = \frac{X_2}{Z_2} \text{ and } x_3 = \frac{X_3}{Z_3}.$$

These require two inversions and two multiplications.

Rewriting gives

$$y_2 = (x_1 + x_2)\left\{\frac{1}{x_1}[X_3(x_1 + x_2) + y_1] + x_2\right\} + y_1$$

$$= (x_1 + x_2)\left\{\frac{1}{x_1 Z_3}[X_3(x_1 + x_2) + Z_3 y_1] + x_2\right\} + y_1$$

To compute this quantity requires inverting $x_1Z_3$ and $Z_2$ and then 4 multiplies.

Since $x_1Z_3$ costs one multiply, the total is 2 inversions and 5 multiplies.

The Odd Characteristic Case:

$$E : y^2 = x^3 + ax + b$$

$$x_3 = \lambda^2 - x_1 - x_2 \qquad \lambda = \frac{y_2 - y_1}{x_2 - x_1}$$

$$x_3 = \left(\frac{y_2 - y_1}{x_2 - x_1}\right) - x_1 - x_2$$

$$= \frac{y_2^2 - 2y_1y_2 + y_1^2 - (x_1 + x_2)(x_2 - x_1)^2}{(x_2 - x_1)^2}$$

$$= y_2^2 - 2y_1y_2 + y_1^2 - (x_1 + x_2)(x_2^2 - 2x_1x_2 + x_1^2)$$

$$= y_2^2 - 2y_1y_2 - x_1x_2^2 + 2x_1^2x_2 - x_2^3 + 2x_1x_2^2 - x_2x_1^2$$

$$= \frac{ax_2 + b + ax_1 + b - 2y_1y_2 - x_1x_2^2 + x_1^2x_2 + 2x_1x_2^2}{(x_2 - x_1)}$$

Solving to $y_2$ gives:

$$2y_1y_2 = -x_3(x_2 - x_1)^2 + ax_2 + 2b + ax_1 + x_1x_2^2 + x_1^2x_2$$

or

$$y_2 = \frac{1}{2y_1}\left\{-x_3(x_2 - x_1)^2 + a(x_2 + x_1) + 2b + x_1x_2(x_1 + x_2)\right\}$$

$$= \frac{1}{2y_1}\left\{(x_1 + x_2) + (a + x_1x_2) - x_3(x_2 - x_1)^2\right\}$$

Replacing $x_3$ we get:

$$x_2 = \frac{X_2}{Z_2}$$

$$y_2 = \frac{1}{2y_1 Z_3}\left\{Z_3(x_1 + x_2)(a + x_1 x_2) - X_2(x_2 - x_1)^2\right\}$$

Similarly, in the field *Fp*,

$$y^2 = x^3 + ax + b$$
$$x_3 = \lambda^2 - x_1 - x_2$$
$$y_3 = \lambda\,(x_1 - x_3) - y_1$$
$$\lambda = \frac{y_2 - y_1}{x_2 - x_1} \qquad (P2 \neq P3)$$

$$x_3 = \frac{y_2^2 - 2y_1 y_2 + y_1^2 - (x_2 - x_1)^2(x_2 + x_1)}{(x_2 - x_1)^2}$$
$$= \frac{y_2^2 - 2y_1 y_2 + y_1^2 - x_2^3 + x_1^2 x_2 + x_1 x_2^2 - x_1^3}{(x_2 - x_1)^2}$$

$$= \frac{ax_2 + b + ax_1 + b - 2y_1 y_2 + x_1 x_2(x_1 + x_2)}{(x_2 - x_1)^2}$$

$$y_2 = \frac{1}{2y_1}\left[(x_1 + x_2) \bullet (a + x_1 x_2) + 2b - x_3(x_2 - x_1)^2\right]$$

$$x_2 = X_2 \frac{1}{Z_2}, x_3 = X_3 \frac{1}{Z_3}$$

which means inversions plus seven multiplies or equivalently one inversion and thirteen multiplies

$3i+7m=1i+13m$

If we replace $x_3$, we have:

$$x_2 = X_2 \frac{1}{Z_2}$$

$$y_2 = \frac{1}{2y_1 Z_3} \bullet \{Z_3 \bullet [(x_1 + x_2),(a + x_1 x_2) + 2b] - X_3 (x_2 - x_1)^2\}$$

In this case, we have two inversions plus eight multiplies or equivalently one inversion and eleven multiplies.

The present invention is thus generally concerned with an encryption method and system and particularly an elliptic curve encryption method and system in which finite field elements is multiplied in a processor efficient manner. The encryption system can comprise any suitable processor unit such as a suitably programmed general-purpose computer.

Although the invention has been described with reference to certain specific embodiments, various modifications thereof will be apparent to those skilled in the art without departing from the spirit and scope of the invention as outlined in the claims appended hereto.

**THE EMBODIMENTS OF THE INVENTION IN WHICH AN EXCLUSIVE PROPERTY OR PRIVILEGE IS CLAIMED ARE DEFINED AS FOLLOWS:**

1. A method of determining a multiple of a point $P$ on an elliptic curve defined over a field $F_{2^m}$, said method comprising steps of:

(a) representing the number $k$ as a vector of binary digits $k_i$;

(b) forming a pair of points $P_1$ and $P_2$, wherein the point $P_1$ and $P_2$ differ at most by $P$; and

(c) selecting each of said $k_i$ in turn and for each of said $k_i$,

upon said $k_i$ being a one, adding said pair of points $P_1$ and $P_2$ to form a new point $P_1$ and adding said point $P$ to $P_1$ to form a new point $P_2$, said new points replacing said pair of points $P_1$ and $P_2$; or

upon said $k_i$ being a zero, doubling said point $P_1$ to form a new point $P_1$ and adding said point $P$ to form a new point $P_2$, said new points replacing said pair of points $P_1$ and $P_2$, whereby said product $kP$ is obtained from said point $P_1$ in $M$-1 steps and wherein $M$ represents the number of digits in $k$.

2. A method as described in claim 1, said elliptic curve being of the form $y^2 + xy = x^3 + ax^2 + b$ and said field being selected to have elements $A^{2^i} (o \leq i \leq m)$ that constitute a normal basis.

3. A method as described in claim 2, including the step of representing the co-ordinates of a point on said curve as a set of vectors, each vector representing a co-ordinate of said point and having m binary digits, each of which represents the coefficients of $A^{2^i}$ in the normal basis representation of said vector.

4. A method as defined in claim 3, said adding of points $P_1$ and $P_2$ utilises only said x co-ordinates of said points $P_1$, $P_2$, and $P_1$-$P_2$.

5. A method as defined in claim 4, said x co-ordinate of said added points is obtained by computing $x_3 + x_4 = \frac{x_1 x_2}{(x_1 + x_2)^2}$ where $x_1$,$x_2$ are the x coordinates of $P_1$ and $P_2$, $x_3$ is the x coordinate of $P_1 + P_2$ and $x_4$ is the x coordinate of $P_1$-$P_2$.

6. A method as defined in claim 5, including converting said coordinates to projective coordinates.

7. A method as defined in claim 6, said coordinate $x_3$ being obtained by computing $x_3 = x_1^4 + bz_1^4$.

8. A method as defined in claim 4, including computing a y coordinate of said point kP from said x coordinate by utilising an x coordinate of said point (k-1)P and said point kP.

9. A method as defined in claim 8, including computing a y coordinate of said point kP by substituting said x coordinate of kP in said elliptic curve equation..

10. A method of transferring the co-ordinates of a point on an algebraic curve between a pair of correspondents connected by a data communications link comprising the steps of forwarding from one correspondent to another a co-ordinate of said point, providing at said other correspondent parameters of said algebraic curve, and computing at said other correspondent said other co-ordinate from said one co-ordinate and said algebraic curve.

11. A method according to claim 10 including the step of forwarding with said one co-ordinate identifying information of said other co-ordinate and utilising said identifying information and a discriminating function to determine the appropriate value of said other co-ordinate.

12. A method according to claim 11 wherein said identifying information is a digital bit of said other co-ordinate that identifies the appropriate value of said other co-ordinate.

13. A method according to claim 12 wherein said algebraic curve is an elliptic curve of the form $y^2 + xy = x^3 + ax^2 + b$ and said other co-ordinate is determined by solving a quadratic equation to provide two possible values of said other co-ordinate, said identifying information indicating the appropriate one of said values.

14. A method according to claim 13 wherein said identifying information is a digital bit of said other co-ordinate that identifies the appropriate value of said other co-ordinate.

15. A method according to claim 14 wherein said algebraic curve is an elliptic curve of the form $y^2 + xy = x^3 + ax + b$ defined over a finite field $F_2^m$.

16. A method according to claim 15 including the step of forwarding with said one co-ordinate identifying information of said other co-ordinate and utilising said identifying information and a discriminating function to determine the appropriate value of said other co-ordinate.

17. A method according to claim 16 wherein said field $GF2^m$ has field elements $A^{2^i}$ that constitute a normal basis.

18. A method according to claim 17 wherein said other co-ordinate is determined by solving a quadratic equation to provide two possible values of said other co-ordinate, said identifying information indicating the appropriate one of said values.

19. A method according to claim 18 wherein said quadratic equation is solved by summing terms of the form $c^{2^g}$ from g = 0 to g = m-1 where $c = X_0 + a + \frac{b}{X_0^2}$ and $x_0$ is said one co-ordinate.

20. A method according to claim 19 wherein terms of the form $c^{2^g}$ are obtained by g fold cyclic shifts of the normal basis representation of c.

21. A method according to claim 20 wherein said algebraic curve is defined over the field **Z**p and said identifying information indicates the Legend symbol of the appropriate value.

22. A method according to claim 21 wherein said curve is defined over the field zp and the elements thereof subdivided into a pair of subsets, one of which contains one

possible value and the other of which contains the other possible value, said indicating information identifying the subset containing the appropriate value.

23. A method for determining a multiple $k$ of a point $P$ having co-ordinates $(x,y)$ on an elliptic curve, said method comprising the steps of:

(a) $x$-co-ordinate of the sum of two points $P_2$ and $P_1$, whose difference is known;

(b) computing the $x$-co-ordinate $x_3$ of a sum of points $P_1$ and $P_2$ from the $x$-co-ordinates of $P$, $P_1$ and $P_2$, wherein the difference between said points $P_1$ and $P_2$ is $P$;

(c) applying said computation over k iterations to obtain the $x$-co-ordinates of a pair of points $kP$ and $(k+1)P$;

(d) using said x-co-ordinate of $kP$ and $(k+1)P$ to obtain a $y$-co-ordinate of $kP$.

24. A method as defined in claim 23, said elliptic curve being defined over a field $F_2m$.

25. A method as defined in claim 23, said elliptic curve being defined over a field $Fp$.

26. A method as defined in claim 24, said y-co-ordinate being obtained by using affine co-ordinates of $kP$ and $(k+1)P$.

27. A method as defined in claim 24, said $y$-co-ordinate being obtained by computing

$$= (x_1 + x_2)\left\{\frac{1}{x_1 Z_3}[X_3(x_1 + x_2) + Z_3 y_1] + x_2\right\} + y_1$$

where $x_3$ is the $x$-co-ordinate of $(k+1)P$, and $x_2$ is the $x$-co-ordinate of $kP$, and $x_2 = \frac{X_2}{Z_2}$

and $x_3 = \frac{X_3}{Z_3}$.

28. A method as defined in claim 25, said $y$-co-ordinate being obtained by computing

$$y_2 = \frac{1}{2y_1 Z_3} \bullet \left\{Z_3 \bullet [(x_1 + x_2), (a + x_1 x_2) + 2b] - X_3 (x_2 - x_1)^2\right\}$$

where $x_3$ is the $x$-co-ordinate of $(k+1)P$, and $x_2$ is the $x$-co-ordinate of $kP$, and $x_2 = \frac{X_2}{Z_2}$ and $x_3 = \frac{X_3}{Z_3}$.

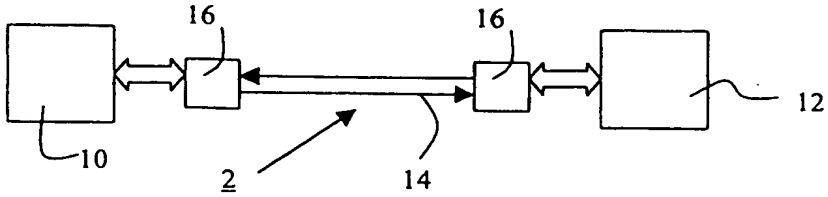

16
16
12
10
2
14

FIG. 1

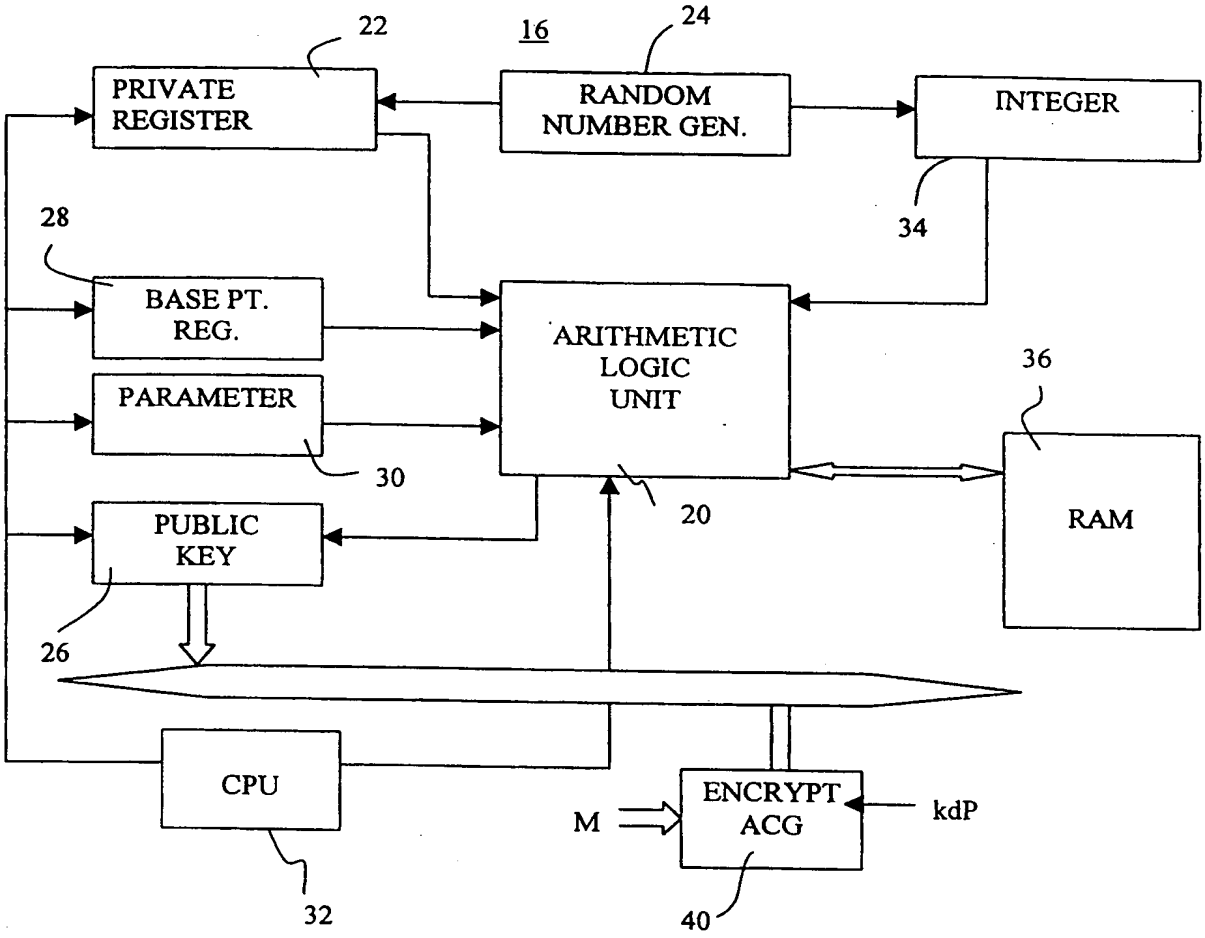

22
16
24
PRIVATE REGISTER
RANDOM NUMBER GEN.
INTEGER
34
28
BASE PT. REG.
ARITHMETIC LOGIC UNIT
36
PARAMETER
30
20
RAM
PUBLIC KEY
26
CPU
M
ENCRYPT ACG
kdP
32
40

FIG. 2

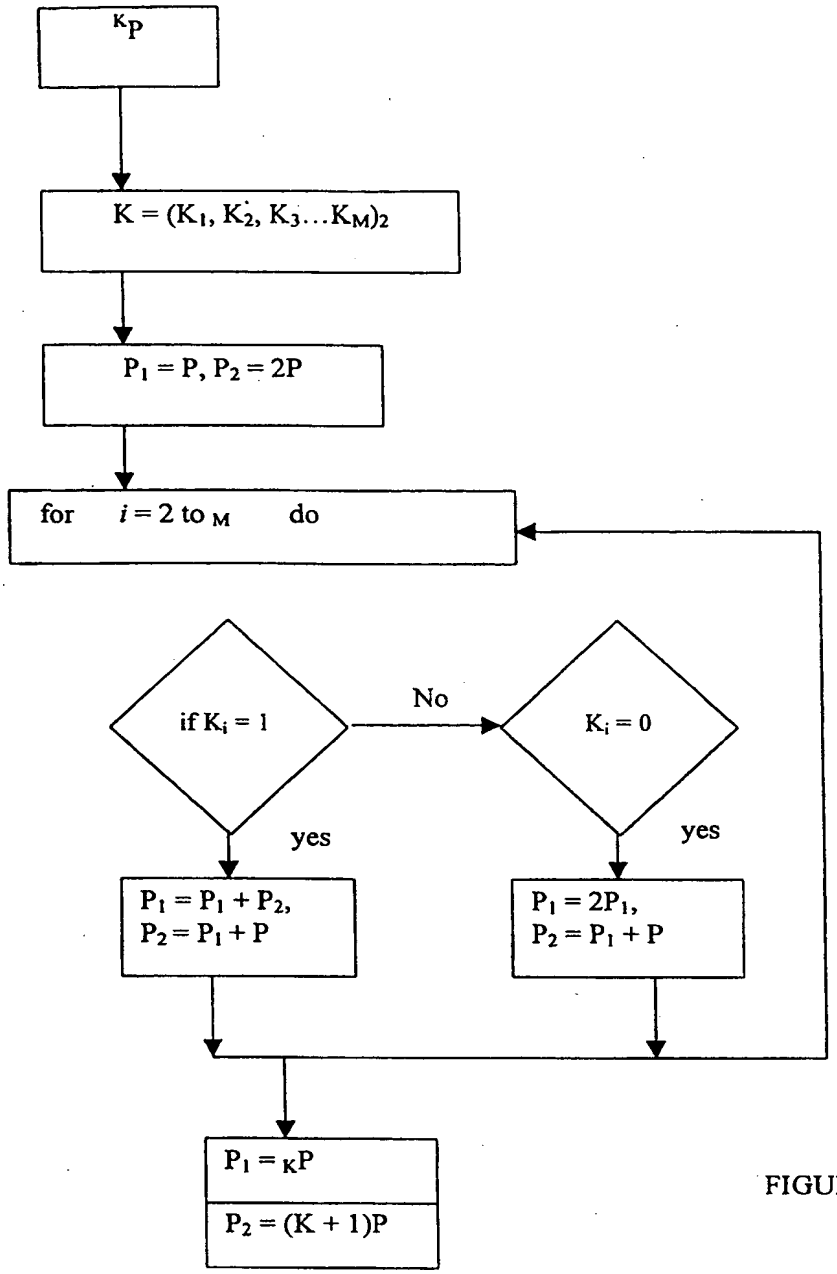

$^{K}P$
$K = (K_1, K_2, K_3 \ldots K_M)_2$
$P_1 = P, P_2 = 2P$
for $i = 2$ to $_M$ do
if $K_i = 1$
No
$K_i = 0$
yes
yes
$P_1 = P_1 + P_2,$
$P_2 = P_1 + P$
$P_1 = 2P_1,$
$P_2 = P_1 + P$
$P_1 = {}_KP$
$P_2 = (K + 1)P$

FIGURE 3

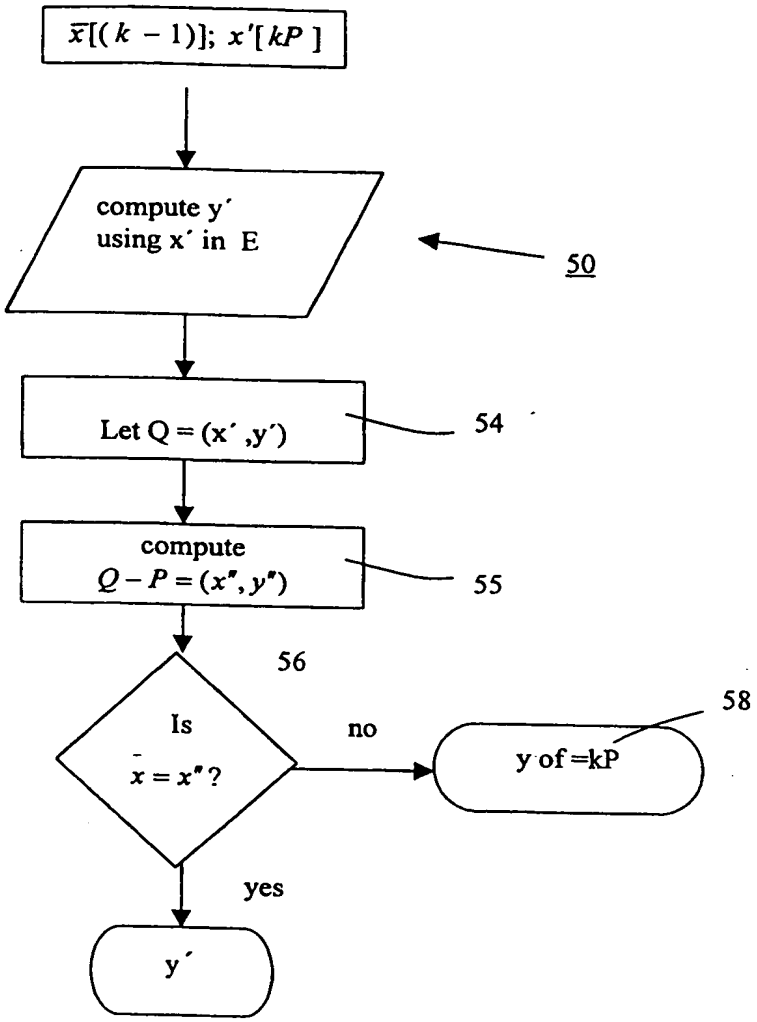

$\bar{x}[(k-1)]; x'[kP]$
compute y'
using x' in E
50
Let Q = (x' ,y')
54
compute
$Q - P = (x'', y'')$
55
56
Is
$\bar{x} = x''$?
no
58
y of =kP
yes
y'

Figure 4

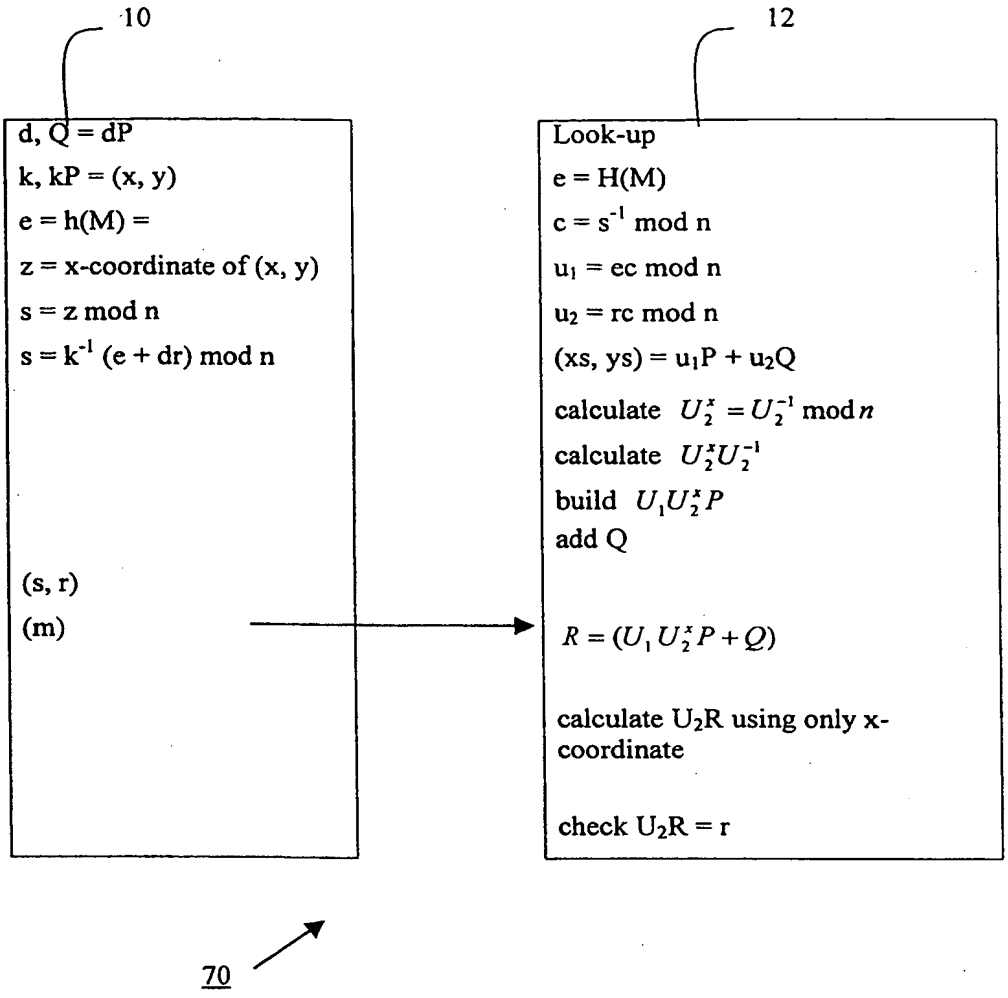

10
12
d, Q = dP
k, kP = (x, y)
e = h(M) =
z = x-coordinate of (x, y)
s = z mod n
s = k^-1 (e + dr) mod n
(s, r)
(m)
Look-up
e = H(M)
c = s^-1 mod n
u1 = ec mod n
u2 = rc mod n
(xs, ys) = u1P + u2Q
calculate U2^x = U2^-1 mod n
calculate U2^x U2^-1
build U1 U2^x P
add Q
R = (U1 U2^x P + Q)
calculate U2R using only x-coordinate
check U2R = r
70

Figure 5

# INTERNATIONAL SEARCH REPORT

Inter onal Application No
PCT/CA 99/00254

A. CLASSIFICATION OF SUBJECT MATTER
IPC 6 G06F7/72

According to International Patent Classification (IPC) or to both national classification and IPC

B. FIELDS SEARCHED

Minimum documentation searched (classification system followed by classification symbols)
IPC 6 G06F H04L

Documentation searched other than minimum documentation to the extent that such documents are included in the fields searched

Electronic data base consulted during the international search (name of data base and, where practical, search terms used)

C. DOCUMENTS CONSIDERED TO BE RELEVANT

| Category ' | Citation of document, with indication, where appropriate, of the relevant passages | Relevant to claim No. |
|---|---|---|
| X,P | EP 0 874 307 A (CERTICOM CORP)<br>28 October 1998 (1998-10-28)<br>the whole document<br>--- | 1-26 |
| X | MENEZES A J ET AL: "ELLIPTIC CURVE CRYPTOSYSTEMS AND THEIR IMPLEMENTATION"<br>JOURNAL OF CRYPTOLOGY,<br>vol. 6, no. 4,<br>1 January 1992 (1992-01-01), pages 209-224, XP002069135<br>cited in the application<br>page 217, last paragraph<br>page 220<br>---<br>-/-- | 1-9,<br>23-26 |

[X] Further documents are listed in the continuation of box C.

[X] Patent family members are listed in annex.

' Special categories of cited documents :

"A" document defining the general state of the art which is not considered to be of particular relevance

"E" earlier document but published on or after the international filing date

"L" document which may throw doubts on priority claim(s) or which is cited to establish the publication date of another citation or other special reason (as specified)

"O" document referring to an oral disclosure, use, exhibition or other means

"P" document published prior to the international filing date but later than the priority date claimed

"T" later document published after the international filing date or priority date and not in conflict with the application but cited to understand the principle or theory underlying the invention

"X" document of particular relevance; the claimed invention cannot be considered novel or cannot be considered to involve an inventive step when the document is taken alone

"Y" document of particular relevance; the claimed invention cannot be considered to involve an inventive step when the document is combined with one or more other such documents, such combination being obvious to a person skilled in the art.

"&" document member of the same patent family

| Date of the actual completion of the international search | Date of mailing of the international search report |
|---|---|
| 7 July 1999 | 13/07/1999 |

| Name and mailing address of the ISA | Authorized officer |
|---|---|
| European Patent Office, P.B. 5818 Patentlaan 2<br>NL - 2280 HV Rijswijk<br>Tel. (+31-70) 340-2040, Tx. 31 651 epo nl,<br>Fax: (+31-70) 340-3016 | Verhoof, P |

1

# INTERNATIONAL SEARCH REPORT

Inte onal Application No
PCT/CA 99/00254

C.(Continuation) DOCUMENTS CONSIDERED TO BE RELEVANT

| Category | Citation of document, with indication, where appropriate, of the relevant passages | Relevant to claim No. |
|---|---|---|
| .X | AGNEW G B ET AL: "AN IMPLEMENTATION OF ELLIPTIC CURVE CRYPTOSYSTEMS OVER F2155" IEEE JOURNAL ON SELECTED AREAS IN COMMUNICATIONS, vol. 11, no. 5, 1 June 1993 (1993-06-01), pages 804-813, XP000399849 page 808 | 1-9, 23-26 |
| X | WO 96 04602 A (AGNEW GORDON B ;MULLIN RONALD C (CA); CRYPTECH SYSTEMS INC (CA); V) 15 February 1996 (1996-02-15) page 15, paragraph 3 page 45, last paragraph - page 48, last paragraph | 10-22 |
| T | KENJI KOYAMA ET AL: "A SIGNED BINARY WINDOW METHOD FOR FAST COMPUTING OVER ELLIPTIC CURVES" IEICE TRANSACTIONS ON FUNDAMENTALS OF ELECTRONICS, COMMUNICATIONS AND COMPUTER SCIENCES, vol. E76 - A, no. 1, 1 January 1993 (1993-01-01), pages 55-61, XP000358858 ISSN: 0916-8508 page 59, right-hand column, last paragraph | 26 |

# INTERNATIONAL SEARCH REPORT

...formation on patent family members

Inter ...nal Application No
PCT/CA 99/00254

| Patent document cited in search report | Publication date | Patent family member(s) | Publication date |
|---|---|---|---|
| EP 0874307 A | 28-10-1998 | CA 2233120 A | 25-09-1998 |
| WO 9604602 A | 15-02-1996 | AU 3073595 A | 04-03-1996 |
| | | EP 0804758 A | 05-11-1997 |

THIS PAGE IS MIS